

Non-uniform vortex lattices in inhomogeneous rotating Bose-Einstein condensates

Gordon Baym,^{a,b} C. J. Pethick,^a S. Andrew Gifford,^b and Gentaro Watanabe^{a,c}

^a*NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*

^b*Department of Physics, University of Illinois, 1110 W. Green St., Urbana, IL 61801*

^c*The Institute of Chemical and Physical Research (RIKEN),
2-1 Hirosawa, Wako, Saitama 351-0198, Japan*

(Dated: February 4, 2008)

We derive a general framework, in terms of elastic theory, for describing the distortion of the vortex lattice in a rotating Bose-Einstein condensate at arbitrary rotation speed and determining the dependence of the distortion on the density inhomogeneity of the system. In the rapidly rotating limit, we derive the energetics in terms of Landau levels, including excitation to higher levels; the distortion depends on the excitation of higher levels as well as on the density gradient. As we show, the dominant effect of higher Landau levels in a distorted lattice in equilibrium is simply to renormalize the frequency entering the lowest Landau level condensate wave function – from the transverse trap frequency, ω , to the rotational frequency, Ω , of the system. Finally, we show how the equilibrium lattice distortion emerges from elastohydrodynamic theory for inhomogeneous systems.

PACS numbers: 05.30.Jp, 03.75.Hh, 03.75.Kk

I. INTRODUCTION

Studies of vortices in rapidly rotating Bose-Einstein condensates [1] have enabled one to look in detail at the structure of the vortex lattice. While, to a first approximation the vortices form a uniform triangular lattice in the limit of a large number of vortices, N_v , Refs. [2] and [3] have shown, for slow rotation, that the vortex lattice undergoes a slight distortion, becoming more widely spaced near the edge of the cloud. The distortion observed later by Coddington et al. [4] is in good agreement with theoretical predictions. The distortion expected in the fast rotation – lowest Landau level (LLL) – limit in a harmonic trap was discussed analytically in Ref. [5] and numerically in Ref. [6]. Our aim in this paper is to produce a general framework for calculating vortex lattice distortions at arbitrary rotation rates in terms of the energetics of the distorted lattice.

The distortion of the vortex lattice is driven by the particle-density gradient in the system. Since the energy of a vortex increases with increasing local density, vortices tend to be forced towards regions of lower density. For clarity, we focus on the directions transverse to the rotation axis (\hat{z}), and assume the system to be uniform along the rotation axis; the ready generalization to include structure in the direction of the rotation axis does not change the results conceptually. In Sec. II we give a general discussion of the energy of a non-uniform vortex lattice. This is applied to the case of slow rotation in Sec. III and to the case of rotation rates comparable to the trap frequency in Sec. IV. Finally, in Sec. V we explore the relationship between the present work and elastohydrodynamic theory [7, 8, 9, 10] which describes the long wavelength modes of the vortex lattice. We take $\hbar = 1$ throughout.

II. ENERGETICS OF VORTEX DISPLACEMENTS

We assume that the vortex displacements from a uniform lattice vary slowly in space and are described by a smooth displacement field, $\epsilon(\mathbf{r})$, and that the lattice rotates at angular frequency Ω . We work primarily in the rotating frame, where the total energy, $E' = E - \Omega L_z$, is generally a functional of the independent degrees of freedom – the displacements of the vortices, the smoothed density, $n(\mathbf{r})$, and the smoothed superfluid velocity in the rotating frame, which we denote by \mathbf{v}_R . [We denote the local density, with all its wiggles near the vortices, by $\hat{n}(\mathbf{r})$, and the microscopic superfluid velocity by $\hat{\mathbf{v}}$.] In the final section we consider the energy as a functional of all these variables. However, in equilibrium, the superfluid velocity is determined by the displacement of the vortices by the equation for quantization of circulation. In this case it is most convenient to regard the energy E' , for a slow variation of the lattice distortion, as a functional only of the smoothed density, $n(\mathbf{r})$, and the smoothed vortex density, $n_v(\mathbf{r})$. The important terms, as we shall see, are the first order change in the energy with ϵ , which is present only for a spatially non-uniform system, and the change in the kinetic energy of rotation induced by a lattice distortion, which leads to a term second order in ϵ .

Since a uniform triangular lattice does not minimize E' , the first order variation of E' under a distortion of the vortex lattice is non-vanishing and of the form

$$\delta E' = \int d^2r \, \delta n_v(\mathbf{r}) Q(n, n_v). \quad (1)$$

(The energy Q is the analog for a vortex of the quasiparticle energy in the Landau theory of Fermi liquids.) To first order in ϵ ,

$$\delta n_v(\mathbf{r}) = -n_v^0 \nabla \cdot \epsilon(\mathbf{r}), \quad (2)$$

where $n_v^0 = m\Omega/\pi$ is the vortex density of the uniform lattice. Integrating by parts, we have

$$\delta^{(1)}E' = \int n_v^0 \boldsymbol{\epsilon} \cdot \nabla Q, \quad (3)$$

where we henceforth denote the integration $\int d^2r$ over the transverse directions simply by \int , and to first order in $\boldsymbol{\epsilon}$ we replace the n_v in Q by n_v^0 . (The second order contribution from the dependence of Q on n_v leads to terms of relative order $1/N_v$ [11].) Since $\nabla Q \simeq (\partial Q/\partial n)\nabla n$, and $\partial Q/\partial n$ is generally positive, vortices are driven towards regions with lower density. They are prevented from moving too far, however, by the rotational kinetic energy of the system which increases as they move away from a perfect triangular lattice.

To determine the kinetic energy of flow in terms of $\boldsymbol{\epsilon}$, we start with the equation for quantization of circulation to relate the modification of the flow velocity to the vortex displacements:

$$\oint_{\mathcal{C}(r)} \hat{\mathbf{v}}(\mathbf{r}') \cdot d\boldsymbol{\ell} = \frac{2\pi}{m} N_v(r), \quad (4)$$

where $\hat{\mathbf{v}}$ is the microscopic local velocity in the lab frame, the contour \mathcal{C} is a circle of radius r about the origin, and $N_v(r)$ is the number of vortices contained within the circle. For a uniform lattice with $N_v(r)$ replaced by $\pi n_v^0 r^2$, the flow is solid body with $v_\phi = \Omega r$. Under a local radial displacement of the vortex lattice, $\delta N_v(r) = -2\pi\epsilon_r r n_v^0$ at fixed r . Thus the smooth flow of the fluid in the azimuthal direction is changed by

$$v_{R,\phi} = -\frac{2\pi}{m} n_v^0 \epsilon_r, \quad (5)$$

which is the fluid flow in the rotating frame. An outward displacement of the vortices leads to a slowing down of the rotational flow. The kinetic energy of the smoothed flow in the rotating frame is

$$\begin{aligned} K &= \frac{1}{2} \int n(r) m v^2 - \Omega \int n(r) m r v_\phi \\ &= \frac{1}{2} \int n(r) m (\mathbf{v} - r\Omega \hat{\phi})^2 - \frac{1}{2} \bar{I} \Omega^2, \end{aligned} \quad (6)$$

where the integral in the second term of the first line is the angular momentum of the flow (the energy associated with the variations about smooth flow is included in Q); and $\bar{I} = \int m n(r) r^2$ is the moment of inertia calculated for the smoothed density profile. A distortion of the vortex lattice at fixed $n(r)$, leads to a second order contribution to the kinetic energy

$$\delta K = \frac{1}{2} \int m n(r) v_{R,\phi}^2, \quad (7)$$

which, with (5), becomes the second order contribution to E' in terms of $\boldsymbol{\epsilon}$:

$$\delta^{(2)}E' = \int \frac{2\pi^2}{m} (n_v^0)^2 n(r) \epsilon_r^2. \quad (8)$$

Corrections to this result are of relative order $1/N_v$.

The energy to second order also includes contributions arising from gradients of $\boldsymbol{\epsilon}$, and this has the usual form for an elastic medium,

$$E_{\text{el}}(r) = \int \left\{ 2C_1 (\nabla \cdot \boldsymbol{\epsilon})^2 + C_2 \left[\left(\frac{\partial \epsilon_x}{\partial x} - \frac{\partial \epsilon_y}{\partial y} \right)^2 + \left(\frac{\partial \epsilon_x}{\partial y} + \frac{\partial \epsilon_y}{\partial x} \right)^2 \right] \right\}, \quad (9)$$

where $C_1(n)$ is the compressional modulus and $C_2(n)$ the shear modulus of the vortex lattice [13]. Since the C 's are of order Ωn , and $\nabla \epsilon$ is of order ϵ/R , where R is the radius of the system, the elastic energy density is of order $\Omega n \epsilon^2/R^2$. By contrast the kinetic energy density in (8) is of order $m\Omega^2 n \epsilon^2$ (where $\ell = 1/\sqrt{\pi n_v}$ is the characteristic spacing of vortices), a factor $\sim R^2/\ell^2 = N_v$ greater than the elastic energy (9); thus the elastic energy may be neglected in determining the distortion of the lattice.

The two important terms in the change in the energy under a vortex displacement together are

$$\delta E' = \int n_v^0 \boldsymbol{\epsilon} \cdot \nabla Q(n) + \int \frac{2\pi^2}{m} (n_v^0)^2 n(r) \epsilon_r^2. \quad (10)$$

Minimizing $\delta E'$ with respect to ϵ_r we find the equilibrium lattice distortion [14],

$$\epsilon_r = -\frac{1}{4\pi\Omega} \frac{dQ}{dr} \quad (11)$$

As we now show, $Q \sim n/m$ in the slow and fast rotation limits, so that $\epsilon \sim -\ell^2 d \ln n / dr$.

III. SLOW ROTATION

In the slowly rotating limit, the kinetic energy of an individual vortex at position \mathbf{R}_j is approximately $\int_j \hat{n}(\mathbf{r})/2m r^2 \approx \gamma n(R_j)$, where \hat{n} is the microscopic particle density in the region of the vortex, the integration is over the Wigner-Seitz cell of radius ℓ containing the vortex, r is measured from the center of the vortex, $\gamma = (\pi/m) \ln(\ell/\xi)$, ξ is proportional to the vortex core size, and the smoothed density, n , is evaluated at the position of the vortex. Thus the leading term in the energy of the vortices is

$$E_{\text{vort}} = \int n_v(r) n(r) \gamma. \quad (12)$$

To leading logarithms, $Q = n\gamma$ in this limit, and

$$\epsilon_r = -\frac{\ell^2}{4} \ln(\ell/\xi) \frac{d \ln n(r)}{dr}, \quad (13)$$

in agreement with Refs. [2] and [3].

Before turning to the rapid rotation limit, we remark that one can calculate Q in general between the slow rotation and very rapid rotation limits using the approach

of Ref. [11]. The calculations, which we do not do here, are complicated by the presence of explicit non-negligible contributions to the vortex energy that depend on dn/dr as well as d^2n/dr^2 , as well as by the need to include explicitly the local fluid velocities in the intermediate regime.

IV. LOWEST LANDAU LEVEL LIMIT

A Bose-condensed system rotating at frequency close to the transverse trap frequency, ω , can be described by the trial condensate wave function,

$$\Psi(\mathbf{r}) = N^{1/2} h(r) \phi_{\text{LLL}}, \quad (14)$$

where $\phi_{\text{LLL}} = \chi e^{-r^2/2d^2}$ is composed only of lowest Landau levels, with χ a polynomial in $z = x + iy$; $d^2 = 1/m\omega$, and N is the total number of particles. As long as the interaction energy per particle is small compared with $\hbar\omega$, one does not expect the modification h of the LLL wave function, which admixes higher Landau levels, to vary significantly over the scale of the intervortex separation. It is thus a reasonable approximation to assume that h is a real slowly varying function dependent only on r .

We first derive the angular momentum in the state (14),

$$\begin{aligned} \langle L \rangle &= \int \Psi^*(\mathbf{r}) \left(z \frac{\partial \Psi(\mathbf{r})}{\partial z} - z^* \frac{\partial \Psi(\mathbf{r})}{\partial z^*} \right) \\ &= N \int d^2r \chi^* h^2 e^{-r^2/d^2} z \frac{\partial \chi}{\partial z}, \end{aligned} \quad (15)$$

where in the latter form we use the fact that $(z\partial/\partial z - z^*\partial/\partial z^*)r^2 = 0$. Then integrating by parts we have

$$\begin{aligned} \langle L \rangle &= -N \int \frac{\partial}{\partial z} \left(z \chi^* h^2 e^{-r^2/d^2} \right) \chi \\ &= -N + \omega I - \int \hat{n} r \frac{d \ln h(r)}{dr}, \end{aligned} \quad (16)$$

where $\hat{n}(\mathbf{r}) = |\Psi(\mathbf{r})|^2$ and $I = m \int \hat{n} r^2$ is the moment of inertia. [The final term in Eq. (16) corrects the expression for $\langle L \rangle$ in Ref. [5].] More generally, for real $h(\mathbf{r}) = h(x, y)$, the expectation value of the angular momentum is

$$\begin{aligned} \langle L \rangle &= \int \hat{n} \left\{ \left(\frac{r^2}{d^2} - 1 \right) - \mathbf{r} \cdot \nabla \ln h(\mathbf{r}) \right\} \\ &= -N + \omega I - \int \hat{n} \mathbf{r} \cdot \nabla \ln h(\mathbf{r}). \end{aligned} \quad (17)$$

To calculate the expectation value of the Hamiltonian in the state (14),

$$E = \int \Psi^* \left(-\frac{\nabla^2}{2m} + \frac{1}{2} m \omega^2 r^2 \right) \Psi + E_{\text{int}}, \quad (18)$$

where E_{int} is the energy due to interparticle interactions, we note that for a lowest Landau level,

$$\begin{aligned} &\left(-\frac{1}{2m} \nabla^2 + \frac{1}{2} m \omega^2 r^2 \right) \chi e^{-r^2/2d^2} \\ &= \omega \left(z \frac{\partial}{\partial z} - z^* \frac{\partial}{\partial z^*} \right) \chi e^{-r^2/2d^2}. \end{aligned} \quad (19)$$

By a calculation similar to that of $\langle L \rangle$ above, we find [15],

$$\begin{aligned} E &= \omega^2 I + \int \frac{\hat{n}}{2m} \left(\frac{d \ln h}{dr} \right)^2 \\ &\quad - \omega \int \hat{n} r \frac{d \ln h(r)}{dr} + E_{\text{int}}. \end{aligned} \quad (20)$$

For a more general $h(\mathbf{r})$, the energy is

$$\begin{aligned} E &= \omega^2 I + \int \frac{\hat{n}}{2m} [\nabla \ln h(\mathbf{r})]^2 - \omega \int \hat{n} \mathbf{r} \cdot \nabla \ln h(\mathbf{r}) \\ &\quad + E_{\text{int}}. \end{aligned} \quad (21)$$

Since $d \ln h(r)/dr$ is of order $1/R$, we may replace $\hat{n}(\mathbf{r})$ by its coarse grained average $n(r)$ in the terms in E and $\langle L \rangle$ involving $d \ln h(r)/dr$. The energy $E' = E - \Omega \langle L \rangle$ in the rotating frame is then

$$\begin{aligned} E' &= \omega(\omega - \Omega) I + \Omega N + \int \frac{n(r)}{2m} \left(\frac{d \ln h}{dr} \right)^2 \\ &\quad - (\omega - \Omega) \int n(r) r \frac{d \ln h(r)}{dr} + \frac{bg}{2} \int n(r)^2, \end{aligned} \quad (22)$$

where b is the Abrikosov parameter.

To determine the structure of the condensate it is most convenient to regard $h(r)$ and the smoothed density $n(r)$ as the independent variables in the energy (22). Variations of $h(r)$ at fixed $n(r)$ must change the local vortex density, $n_v(r)$. To see how, we write the generalization for the state (14) of the result of Ref. [12] relating the particle density to the vortex positions:

$$\frac{1}{4} \nabla^2 \ln \left(\frac{n(r)}{h^2(r)} \right) = -m\omega + \pi n_v(r). \quad (23)$$

Thus, under a variation of $h(r)$ at fixed $n(r)$,

$$\delta n_v(r) = -\frac{1}{2\pi} \delta \nabla^2 \ln h(r). \quad (24)$$

While the moment of inertia, \bar{I} , for the smoothed density is independent of $h(r)$ at fixed $n(r)$, the difference $I - \bar{I}$ does depend on the density of vortices, and, as shown in Ref. [11], has the structure,

$$I - \bar{I} \sim \int \frac{m}{\pi} \frac{n(r)}{n_v(r)}. \quad (25)$$

Thus under a variation of $h(r)$, we find, after integration by parts, that

$$\delta(I - \bar{I}) \sim \int \frac{1}{m\Omega^2} \frac{dn(r)}{dr} \delta \frac{d \ln h}{dr}. \quad (26)$$

This term is of relative order $1/N_v$ compared with the variation of the term $-(\omega - \Omega) \int n(r) r d \ln h(r) / dr$ in E' , and can be neglected.

We now minimize the energy (22) with respect to $h(r)$ at fixed smooth density $n(r)$ and find,

$$\nabla \ln h(r) = m(\omega - \Omega) \mathbf{r}, \quad (27)$$

so that

$$h(r) = C e^{m(\omega - \Omega) r^2 / 2}. \quad (28)$$

Substituting this result back into Eq. (14), we obtain

$$\Psi(\mathbf{r}) \sim N^{1/2} \chi e^{-m\Omega r^2 / 2}. \quad (29)$$

Although in terms of the frequency ω , this wave function includes higher Landau levels via h , their only effect is to change the oscillator frequency in the levels to the rotation frequency, Ω .

Minimization of E' with respect to $n(r)$, with Eq. (28), yields the Thomas-Fermi profile,

$$\begin{aligned} n(r) &= \frac{1}{bg} \left(\mu - \frac{mr^2}{2} (\omega^2 - \Omega^2) \right) \\ &= n(0) \left(1 - \frac{r^2}{R^2} \right) \end{aligned} \quad (30)$$

plus terms of relative order $1/N_v$. Here μ is the chemical potential. The radius of the cloud is $R = [8Nbg/\pi m(\omega^2 - \Omega^2)]^{1/4}$. The Thomas-Fermi profile reflects the reduction of the effective trapping potential by the centrifugal potential.

We now derive the distortion of the lattice. Measuring the displacements with respect to a uniform triangular lattice of vortex density $n_v^0 = m\Omega/\pi$, and using use Eq. (27) in (23), with (2), we find,

$$\frac{1}{4} \nabla^2 \ln \left(\frac{n(r)}{h^2(r)} \right) = -m(\omega - \Omega) - n_v^0 \nabla \cdot \boldsymbol{\epsilon}(r). \quad (31)$$

For equilibrium lattices, the displacement $\boldsymbol{\epsilon}$ is entirely in the radial direction, and we may integrate (31) to find,

$$\frac{d \ln h(r)}{dr} = \frac{1}{2} \frac{d \ln n(r)}{dr} + m(\omega - \Omega) r + 2\pi n_v^0 \epsilon_r. \quad (32)$$

With Eq. (27) for h , this result reduces to

$$\epsilon_r = -\frac{1}{4m\Omega} \frac{d \ln n}{dr}, \quad (33)$$

which has the form (11) with $Q = \pi n/m$. Using the Thomas-Fermi profile, we find

$$\epsilon_r = \frac{r}{2m\Omega} \frac{1}{R^2 - r^2}. \quad (34)$$

Were we to assume a uniform lattice ($\boldsymbol{\epsilon} = 0$), then Eq. (32) would imply

$$h(r) \sim \sqrt{n(r)} e^{m(\omega - \Omega) r^2 / 2}, \quad (35)$$

and thus

$$\Psi \sim \sqrt{n(r)} \chi e^{-m\Omega r^2 / 2}. \quad (36)$$

The admixture of higher Landau levels would not only modify the effective frequency, but modulate the wave function by the square root of the smoothed density. This result for $h(r)$ does not minimize the energy, due to the imposed constraint of a uniform triangular lattice [16].

A. Elastic energy in the LLL

We now compute the terms in the energy associated with the distortion of the vortex lattice. Inserting Eq. (32) for dh/dr into Eq. (22), we derive the energy in the rotating frame, written in terms of the coarse grained density, $n(r)$, and the lattice displacement:

$$\begin{aligned} E'\{n(r), \epsilon_r(r)\} &= \frac{\omega^2 - \Omega^2}{2} \bar{I} + \omega(\omega - \Omega)(I - \bar{I}) + E_{\text{int}} \\ &\quad + \Omega \int \epsilon_r \frac{dn}{dr} + 2m\Omega^2 \int n \epsilon_r^2. \end{aligned} \quad (37)$$

The penultimate term has the same form as that in the slowly rotating limit, (12), only with $\gamma = \pi/m$. The final term is the same as in (10), and is the modification of the kinetic energy associated with the decreased azimuthal velocity caused by radial vortex displacements. Again, minimizing Eq. (37) with respect to ϵ_r , dropping the corrections arising from the difference between I and \bar{I} , which are suppressed by a factor $1 - \Omega/\omega$, we are led directly to (33). The angular momentum in terms of ϵ_r and n is

$$\langle L \rangle = \omega(I - \bar{I}) + \Omega \bar{I} - 2m\Omega \int n r \epsilon_r. \quad (38)$$

V. CONNECTION WITH ELASTOHYDRODYNAMICS

The results for the equilibrium lattice distortion are all contained in the elastohydrodynamic theory of Refs. [7, 8, 9, 10], when the dependence of the elastic energy on density gradients is taken into account. The key equations in the elastohydrodynamic theory (all in the rotating frame, denoted by a subscript R) are the conservation of circulation,

$$\mathbf{v}_R + 2\boldsymbol{\Omega} \times \boldsymbol{\epsilon} = \nabla \Phi / m, \quad (39)$$

where Φ is the superfluid phase; the superfluid acceleration equation, the time derivative of Eq. (39),

$$m \left(\frac{\partial \mathbf{v}_R}{\partial t} + 2\boldsymbol{\Omega} \times \dot{\boldsymbol{\epsilon}} \right) = -\nabla(\mu + V_{\text{eff}}), \quad (40)$$

where μ is the chemical potential of the matter; and the equation for conservation of momentum,

$$m \frac{\partial \mathbf{j}_R}{\partial t} + 2m\boldsymbol{\Omega} \times \mathbf{j}_R + \nabla P + n \nabla V_{\text{eff}} = -\boldsymbol{\sigma}. \quad (41)$$

Here \mathbf{v}_R and \mathbf{j}_R are the smoothed superfluid velocity and current density in the rotating frame, P is the pressure, and $V_{\text{eff}}(r) = V_{\text{trap}}(r) - \frac{1}{2}m\Omega^2 r^2$. The elastic force, $-\boldsymbol{\sigma}$, is given by

$$\boldsymbol{\sigma}(\mathbf{r}, t) = \left. \frac{\delta E'}{\delta \boldsymbol{\epsilon}} \right|_{n, \mathbf{v}_R}, \quad (42)$$

where E' is the full energy in the rotating frame for the distorted lattice, including the usual elastic energy (9). In the elastohydrodynamics, which also describes non-equilibrium dynamics, the smoothed density, $n(\mathbf{r})$, the displacement, $\boldsymbol{\epsilon}(\mathbf{r})$, and the smoothed superfluid velocity, $\mathbf{v}_R(\mathbf{r})$, are independent dynamical degrees of freedom, and one must regard E' as a functional of these three independent variables. Only in the determination of $\boldsymbol{\epsilon}$ in equilibrium can one eliminate the superfluid velocity as an independent variable. Keeping \mathbf{v}_R fixed means that the kinetic energy variation, δK , and hence $\delta^{(2)}E'$, Eqs. (7) and (8), do not contribute to $\boldsymbol{\sigma}$.

In equilibrium, where one has axial symmetry, Eq. (39) reduces simply to (5); then from Eq. (40), $d(\mu + V_{\text{eff}}) = 0$. In addition, at zero temperature, $\nabla P = n \nabla \mu$.

To show how the displacement of the equilibrium lattice, (33), emerges from the elastohydrodynamics, we write $\mathbf{j}_R = n\mathbf{v}_R$ and subtract Eq. (40) from (41) divided by n , to find [17],

$$2m\boldsymbol{\Omega} \times (\mathbf{v}_R - \dot{\boldsymbol{\epsilon}}) = -\frac{\boldsymbol{\sigma}}{n}. \quad (43)$$

The radial component of this equation in equilibrium reads

$$2m\Omega v_{R,\phi} = \frac{\sigma_r}{n}, \quad (44)$$

so that from Eq. (5),

$$4m\Omega^2 \epsilon_r = -\frac{\sigma_r}{n}. \quad (45)$$

Evaluating σ from the elastic energy contribution $\Omega \int \epsilon_r dn/dr$ in Eq. (37) we are led immediately to the vortex displacement (33) in the LLL.

Acknowledgments

Author GB is grateful for the hospitality of the Aspen Center for Physics, where part of this work was carried out. Author GW was supported in part by Grants-in-Aid for Scientific Research provided by the Ministry of Education, Culture, Sports, Science, and Technology through Research Grant No. 14-7939, by the Nishina Memorial Foundation, and by a JSPS Postdoctoral Fellowship for Research Abroad. This research was also supported in part by NSF Grants PHY03-55014 and PHY05-00914.

-
- [1] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000); F. Chevy, K. W. Madison, and J. Dalibard, Phys. Rev. Lett. **85**, 2223 (2000); K. W. Madison, F. Chevy, V. Bretin, and J. Dalibard, Phys. Rev. Lett. **86**, 4443 (2001). J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle, Science **292**, 476 (2001). P. C. Haljan, I. Coddington, P. Engels, and E. A. Cornell, Phys. Rev. Lett. **87**, 210403 (2001); P. Engels, I. Coddington, P. C. Haljan, and E. A. Cornell, Phys. Rev. Lett. **89**, 100403 (2002). V. Schweikhard, I. Coddington, P. Engels, V. P. Mogendorff, and E. A. Cornell, Phys. Rev. Lett. **92**, 040404 (2004).
 - [2] J. R. Anglin and M. Crescimanno, cond-mat/0210063; also, J.R. Anglin, Phys. Rev. A **65**, 063611 (2002).
 - [3] D. E. Sheehy and L. Radzihovsky, Phys. Rev. A **70**, 051602(R) (2004); *ibid.*, 063620 (2004).
 - [4] I. Coddington, P. C. Haljan, P. Engels, V. Schweikhard, S. Tung, and E. A. Cornell, Phys. Rev. A **70**, 063607 (2004).
 - [5] G. Watanabe, G. Baym, and C. J. Pethick, Phys. Rev. Lett. **93**, 190401 (2004).
 - [6] N. R. Cooper, S. Komineas, and N. Read, Phys. Rev. A **70**, 033604 (2004); see also the qualitative discussion in E. B. Sonin Phys. Rev. A **72**, 021606(R) (2005).
 - [7] G. Baym and E. Chandler, J. Low Temp. Phys. **50**, 57 (1983); E. Chandler and G. Baym, **62**, 119 (1986).
 - [8] G. Baym, Phys. Rev. Lett. **91**, 110402 (2003).
 - [9] G. Baym, Phys. Rev. A **69**, 043618 (2004).
 - [10] S. A. Gifford and G. Baym, Phys. Rev. A **70**, 033602 (2004).
 - [11] G. Watanabe, S. A. Gifford, G. Baym, and C. J. Pethick, cond-mat/0606151.
 - [12] T.-L. Ho, Phys. Rev. Lett. **87**, 060403 (2001).
 - [13] For a purely radial vortex displacement, the elastic energy reduces to,

$$E_{\text{el}}(r) = 2C_1 \left[\frac{1}{r} \frac{d(r\epsilon_r)}{dr} \right]^2 + C_2 \left[r \frac{d(\epsilon_r/r)}{dr} \right]^2.$$
 The C_2 term vanishes for homologous displacements, $\epsilon_r \propto r$.
 - [14] Similar arguments on the relation of the vortex density and the inhomogeneity of the density profile have been given by J. Anglin (private communication); also N. Read (private communication).
 - [15] The calculations of Ref. [5] inadvertently omitted the term proportional to $d \ln h/dr$.
 - [16] Furthermore, if the vortex density in the uniform lattice is $n_v^0 = m\Omega_v/\pi$, (32) would imply: $\nabla \ln h(r) = \nabla \ln \sqrt{n(r)} + m(\omega - \Omega_v)\mathbf{r}$. Substitution of this result into (22) would then give

$$E' = N\Omega_v + \int \frac{(\nabla \sqrt{n})^2}{2m} + E_{\text{int}} + \frac{1}{2} \bar{I}(\omega^2 + \Omega_v^2 - 2\Omega\Omega_v) + \omega(\omega - \Omega)(I - \bar{I}).$$

This result agrees, to within terms of relative order $1/N_v$, with that derived in [11] by considering the energies of the individual Wigner-Seitz cells containing the vortices.

- [17] To derive the conservation of energy, to linear order, we multiply Eq. (41) by \mathbf{v}_R . After some manipulation and use of the orthogonality of $\delta E'/\delta \epsilon|_{n, \mathbf{v}_R}$ to $\mathbf{v}_R - \dot{\epsilon}$, seen from Eq. (43), the equation of energy conservation be-

comes,

$$\frac{\partial}{\partial t} \frac{mnv_R^2}{2} + \dot{n} \left. \frac{\partial E'}{\partial n} \right|_{\mathbf{v}_R, \epsilon} + \dot{\epsilon} \cdot \left. \frac{\delta E'}{\delta \epsilon} \right|_{n, \mathbf{v}_R} + \nabla \cdot \mathbf{j}_E = 0,$$

where \mathbf{j}_E is the energy current in the rotating frame.